

EQUIVARIANT COHOMOLOGY AND TENSOR CATEGORIES

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ABSTRACT. In this research announcement we propose the notion of a *supercategory* as an alternative approach to supermathematics. We show that this setting is rich to carry out many of the basic constructions of supermathematics. We also prove generalizations of a number of results in equivariant cohomology, including the Chern-Weil theorem for an arbitrary rigid Lie algebra object. For a quadratic Lie algebra object we obtain a proof of the Duflo isomorphism along the lines of Alekseev-Meinrenken, thereby generalizing their result to Lie superalgebras.

1. INTRODUCTION

As pointed out in [De-Mo] many constructions of multilinear algebra and differential geometry can be carried out in the context of a tensor category, and the more general perspective often yields additional insights. One advantage of this approach is that it treats vector spaces and super vector spaces on an equal footing, thus avoiding the signs which proliferate in the former setting. However one sometimes needs the signs in order to deal with essentially “super” concepts, such as that of a connection.

In this research announcement we propose the notion of a *supercategory* as an alternative approach to super-mathematics. We show that this setting is rich enough to carry out many of the basic constructions; and to further demonstrate its power we establish a number of results in equivariant cohomology, including the Chern-Weil theorem for an arbitrary rigid Lie algebra object. In the case of a quadratic Lie algebra object we obtain a proof of the Duflo isomorphism (see [Duf]) along the lines of [Al-Me 2]. Our result implies in particular a generalization of the Alekseev-Meinrenken result to the setting of Lie superalgebras. The Duflo isomorphism for Lie algebras has been generalized in the Kashiwara-Vergne Conjecture (now Theorem : see [KV], [Ver], [Tor] and the references therein). It would be interesting to see whether this admits an extension to supercategories.

The proofs will appear elsewhere.

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1.1. Supercategories. A tensor category is an abelian category equipped with a bilinear product \otimes , which is associative, commutative and has a unit I . We will write XY for $X \otimes Y$, and $X(\cdot)$ for the functor $Y \mapsto XY$. A tensor category is said to be \mathbb{Q} -linear if there is an inclusion $\mathbb{Q} \subset \text{End}(I)$ (this implies that all the hom-sets are \mathbb{Q} -vector spaces). For basic facts about tensor categories, including definitions of tensor, symmetric and exterior powers $T^*(X), S^*(X), \Lambda^*(X)$, we refer the reader to [De-Mo].

Definition 1.1. *A supercategory is a well-powered and complete \mathbb{Q} -linear tensor category \mathcal{S} , equipped with the choice of an object P satisfying $\Lambda^2(P) \approx I, S^2(P) \approx 0$.*

As developed in [De-Mo, p.45], an important example of a supercategory is the category of superspaces over a field k of characteristic 0. The category admits a *parity reversal functor* Π . Objects in a general supercategory need not be bigraded. However we still have a parity reversal functor, viz. $P(\cdot)$, which satisfies $P^2X \approx X$. One can also define the concept of an *odd morphism* from X to Y , an element of

$$(1) \quad \text{Hom}^{\text{odd}}(X, Y) := \text{Hom}(PX, Y) \approx \text{Hom}(X, PY)$$

where the two ordinary hom-spaces are identified via P . For $f \in \text{Hom}^{\text{odd}}(X, Y)$ and $g \in \text{Hom}^{\text{odd}}(Y, Z)$, their composition gf is an ordinary morphism in $\text{Hom}(X, Z) \approx \text{Hom}(PX, PZ)$.

1.2. Algebras and modules. An “ordinary” algebra or module consists of a vector space over a field k , with some algebraic structure which can be formulated in terms of linear maps. Thus the definitions continue to make sense if we replace the category of k -vector spaces by a tensor category. In particular for a supercategory \mathcal{S} we can define categories $\text{Asso}(\mathcal{S}), \text{Com}(\mathcal{S}), \text{Lie}(\mathcal{S})$ of associative, commutative and Lie algebras, respectively.

Objects in these algebra categories have structure morphisms: 1) product/Lie bracket: $A \otimes A \rightarrow A$ and 2) (for $\text{Asso}(\mathcal{S}), \text{Com}(\mathcal{S})$) unit: $I \rightarrow A$. For each algebra A , one can also define a category $\text{Mod}(A)$ of A -modules; these consist of an underlying object M in \mathcal{S} and a morphism $A \otimes M \xrightarrow{a} M$ called “action”, required to satisfy the usual conditions, formulated in categorical terms.

Proposition 1.2. *If $L \in \text{Lie}(\mathcal{S})$ then $\text{Mod}(L)$ is a supercategory with a forgetful functor F_L to \mathcal{S} . There is a trivial extension functor $E_L : \mathcal{S} \rightarrow \text{Mod}(L)$ where for every X in \mathcal{S} the L -action on $E_L(X)$ is 0. Furthermore, E_L admits a right adjoint $M \mapsto M^L : \text{Mod}(L) \rightarrow \mathcal{S}$, called the “invariants functor”.*

This proposition underlies an important philosophical point in our approach : L -equivariant analogs of various algebraic constructions can be regarded as “ordinary” constructions in the supercategory $\text{Mod}(L)$, and this point of view leads automatically to the correct definitions.

2. COHOMOLOGY

2.1. The category \mathcal{Q} and cohomology. We fix a Lie algebra Q whose underlying object is isomorphic to P , and which is abelian, i.e., the bracket $Q \times Q \rightarrow Q$ is the 0 morphism. One sees easily that the supercategory $\mathcal{Q} = \text{Mod}(Q)$ is naturally isomorphic to the category of “supercomplexes”. It consists of pairs (X, d) , where X is an object in \mathcal{S} and d (the “differential”) is an odd endomorphism $d \in \text{Hom}^{\text{odd}}(X, X)$ satisfying $d^2 = 0$. We may therefore define a functor $\mathcal{Q} \rightarrow \mathcal{S}$, the *\mathcal{Q} -cohomology functor*, as follows:

$$(2) \quad H^{\mathcal{Q}}(X) = (\ker d)/(\text{im } d).$$

Since the second and higher symmetric powers of P vanish, the symmetric algebra $D = S(P)$ has underlying object $I \oplus P$. In fact D has a natural commutative \mathcal{Q} -algebra structure with differential d given as follows: $I \xrightarrow{d} 0$, while $P \xrightarrow{d} I$ is the odd endomorphism corresponding to the identity. Tensoring with D defines the *doubling functor* from \mathcal{S} to \mathcal{Q} , which takes \mathcal{S} -algebras to \mathcal{Q} -algebras and A -modules to DA -modules.

2.2. \mathcal{L} -cohomology and homotopy. For the rest of the note we fix a Lie algebra $L_0 \in \text{Lie}(\mathcal{S})$. We write L for $DL_0 \in \text{Lie}(\mathcal{Q})$ and \mathcal{L} for the supercategory $\text{Mod}(L)$. We define the \mathcal{L} -cohomology functor $H^{\mathcal{L}}(X) := H^{\mathcal{Q}}(X^L)$ from $\mathcal{L} \rightarrow \mathcal{S}$, and for any $Z \in \mathcal{L}$ we define the twisted cohomology functor $H_Z^{\mathcal{L}}(X) := H^{\mathcal{L}}(ZX)$. If L_0 is understood we will simply write H and H_Z for the two functors.

A *homotopy* between two \mathcal{L} -morphisms $f, g : X \rightarrow Y$ is an odd \mathcal{S} -morphism $h : X \rightarrow Y$ which commutes with the L -action and satisfies $dh + hd = f - g$. An \mathcal{L} -morphism $f' : Y \rightarrow X$ is said to be the *homotopy inverse* of $f : X \rightarrow Y$ if ff' and $f'f$ are homotopic to the identity morphisms for Y and X respectively.

Proposition 2.1. *Suppose $f : A \rightarrow A'$ is an algebra morphism in $\text{Asso}(\mathcal{L})$, B is an algebra in $\text{Asso}(\mathcal{L})$, and C is a B -module; then*

- (1) $H_A(B)$ is an algebra in $\text{Asso}(\mathcal{S})$ and $H_A(C)$ is an $H_A(B)$ module.
- (2) f induces an algebra morphism $\tau_B(f) : H_A(B) \rightarrow H_{A'}(B)$ and a compatible module morphism $\tau_C(f) : H_A(C) \rightarrow H_{A'}(C)$.
- (3) If g is homotopic to f then $\tau_B(f) = \tau_B(g)$ and $\tau_C(f) = \tau_C(g)$.

2.3. Quasi-isomorphism. An algebra morphism in $\text{Asso}(\mathcal{L})$ is said to be a *quasi-isomorphism* if it admits a homotopy inverse as an \mathcal{L} -morphism. As it turns out, many algebra morphisms defined by universal properties turn out to be quasi-isomorphisms, emphasizing the importance of :

Proposition 2.2. *Suppose $f : A \rightarrow A'$ is a quasi-isomorphism.*

- (1) *If B is an algebra in \mathcal{L} then $\tau_B(f) : H_A(B) \rightarrow H_{A'}(B)$ is an algebra isomorphism.*
- (2) *If C is a B -module, then $\tau_C(f) : H_A(C) \rightarrow H_{A'}(C)$ is a module isomorphism.*

3. CHERN-WEIL THEORY

3.1. Connections. Suppose G is a compact Lie group with Lie algebra \mathfrak{g} and E is a principal G -bundle. A (principal) connection on E is a G -map θ from \mathfrak{g}^* to $\Omega(E)$, the de Rham complex of differential forms on E , satisfying certain properties. This definition can be generalized to the present situation, but in order to carry this out we need the concepts of “dual object” and “unital object”.

In a tensor category, an object Y is said to be *dual* to X if there exist compatible morphisms $I \rightarrow X \otimes Y$ (inclusion) and $Y \otimes X \rightarrow I$ (evaluation). An object X is *rigid* if it has a dual, which is then unique. If so, X is reflexive. For the category of vector spaces, rigidity corresponds to finite dimensionality.

A *unital object* is an object X in \mathcal{S} together with a distinguished morphism $u : I \rightarrow X$. There is an obvious notion of unital morphisms between unital objects and hence unital objects in \mathcal{S} form a category $\text{Unit}(\mathcal{S})$. Moreover, there is a natural sequence of forgetful functors $\text{Com}(\mathcal{S}) \rightarrow \text{Asso}(\mathcal{S}) \rightarrow \text{Unit}(\mathcal{S}) \rightarrow \mathcal{S}$.

Definition 3.1. Assume that L_0 is rigid. A connection on an object X in $\text{Unit}(\mathcal{L})$ is a $\theta \in \text{Hom}^{\text{odd}}(L_0^*, X)$ for which the diagram commutes (the vertical maps are the evaluation and the L action on X , respectively):

$$(3) \quad \begin{array}{ccc} PL_0 \otimes PL_0^* & \xrightarrow{1 \otimes \theta} & PL_0 \otimes X \\ \downarrow & \circlearrowleft & \downarrow \\ I & \xrightarrow{u} & X. \end{array}$$

3.2. Weil objects. For a unital object X , we write $\Theta(X)$ for the set of connections on X . Then Θ is a functor from $\text{Unit}(\mathcal{L})$ (therefore also from $\text{Asso}(\mathcal{L}), \text{Com}(\mathcal{L})$) to Set .

Theorem 3.2. Θ is a representable functor from $\text{Unit}(\mathcal{L}), \text{Asso}(\mathcal{L}), \text{Com}(\mathcal{L})$ to Set . The representing objects, the Weil objects, are defined as objects in \mathcal{S} (and as L_0 modules) by

$$(4) \quad M_L \sim I \oplus DL_0^*, \quad A_L = T_1(M_L) \sim T(DL_0^*), \quad C_L = S_1(M_L) \sim S(DL_0^*)$$

For any connection θ on an object X in any of the three categories, write c_θ for the corresponding “Chern-Weil” morphism from M_L, A_L, C_L to X . We have the following analog of [Car1, théorème 3].

Theorem 3.3. For any two connections θ, θ' on X , the Chern-Weil morphisms $c_\theta, c_{\theta'}$ are homotopic.

Following again [Car1], we define the *equivariant cohomology functor* as $H_{\text{eq}} = H_{C_L}$. Then we have the following result which generalizes the computation of the equivariant cohomology of a point.

Proposition 3.4. $H_{\text{eq}}(I) = H(C_L) \approx S(L_0^*)^{L_0}$

3.3. Main results. Our main results are three quasi-isomorphism theorems. The first one requires only rigidity for L_0 .

Theorem 3.5. *The map $A_L \rightarrow C_L$ is a quasi-isomorphism.*

Assume now that L_0 is a *quadratic* Lie algebra, i.e., there is an L_0 -module isomorphism $L_0^* \approx L_0$. Then the unital Weil object M_L is naturally a *unital Lie algebra*, i.e. it admits a distinguished Lie algebra morphism from the trivial Lie algebra I . In fact one has the following central extension of Lie algebras: $0 \rightarrow I \rightarrow M_L \rightarrow L \rightarrow 0$. Now for any unital Lie algebra one can define its *unital enveloping algebra*, the universal object corresponding to the forgetful functor from associative algebras to unital Lie algebras. We write E_L for the unital enveloping algebra of M_L . We obtain a natural algebra morphism $A_L \rightarrow E_L$:

Theorem 3.6. *The map $A_L \rightarrow E_L$ is a quasi-isomorphism.*

Following [Al-Me 2] we can define noncommutative and quantized equivariant cohomology (the latter for quadratic Lie algebras) : $\mathbf{H}_{\text{eq}} = H_{A_L}$ and $\mathcal{H}_{\text{eq}} = H_{E_L}$.

Corollary 3.7. *For any algebra B in $\text{Asso}(\mathcal{L})$, the algebras $H_{\text{eq}}(B)$, $\mathbf{H}_{\text{eq}}(B)$ and $\mathcal{H}_{\text{eq}}(B)$ are isomorphic.*

Proposition 3.8. $H(E_L) \approx U(L_0)^{L_0}$.

Combining Propositions 3.4, 3.8 and Theorems 2.2, 3.5, 3.6, we obtain for L_0 quadratic (cf [Duf]) :

Corollary 3.9 (Duflo isomorphism). *The algebras $U(L_0)^{L_0}$ and $S(L_0)^{L_0}$ are isomorphic.*

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